GENERATION OF HARMONIC TEMPERATURE VARIATIONS

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An analytic method is presented for calculating the parameters of a thermopile — a source of harmonic surface-temperature fluctuations.

The possibility of reproducing a specified temperature variation of the working surface of a thermopile by controlling the supply current has been studied experimentally and theoretically [1, 2]. In particular, an equation was derived in [2] relating the variation of the supply current to the specified temperature variation, and the necessary or sufficient conditions for reproducing the temperature variation were indicated.

When the temperature $\Theta^{\circ}(\mathrm{Fo})$ is required to vary periodically, the equation for the current $\nu(\mathrm{Fo})$ has the form [2]

$$\nu(\mathrm{Fo}) = [\Theta^{(0)}(\mathrm{Fo})]^{-1}\left\{\xi\nu^2(\mathrm{Fo}) + \int\limits_{-\infty}^{\mathrm{Fo}} K(\mathrm{Fo}-\tau)\nu^2(\tau)\,d\tau + \Phi^*[\mathrm{Fo},\ \Theta^{(0)}(\mathrm{Fo})]\right\}. \tag{1}$$

The kernel of the integral equation (1) is

$$K(\mathrm{Fo}) = 4\sum_{k=1}^{\infty}\exp[-\pi^2(2k-1)^2\mathrm{Fo}]. \tag{2}$$

If the function $\Theta^{\circ}(\mathrm{Fo})$ is such that Eq. (1) has a solution, it is shown in [2] that it is expedient to obtain it by numerical iterations. It is shown below that when the thermopile generates harmonic temperature fluctuations, the form of the current $\nu(\mathrm{Fo})$ can be found analytically.

Let the temperature of the cold junctions vary with time according to the law

$$\Theta^{(0)}(\mathrm{Fo}) = \bar{\Theta} - a\sin\omega\mathrm{Fo}. \tag{3}$$

When $\Theta^{(\circ)}$ varies harmonically the function Φ^* is determined by the equation [2]

$$\Phi^* = A + a[\alpha(\omega)\sin\omega\mathrm{Fo} + \beta(\omega)\cos\omega\mathrm{Fo}]. \tag{4}$$

Here

$$A = \mathrm{Bi}(\Theta_0 - \bar{\Theta}) + \Theta_1 - \bar{\Theta},\ \alpha(\omega) = 1 + \mathrm{Bi} + S(\omega),\ \beta(\omega) = \eta\omega + R(\omega),$$

$$S(\omega) = 2\omega^2\sum_{k=1}^{\infty}\frac{1}{k^4\pi^4+\omega^2},\ R(\omega) = 2\pi^2\omega\sum_{k=1}^{\infty}\frac{k^2}{k^4\pi^4+\omega^2},$$

θ_0 and θ_1 are the ambient temperature and the temperature of the hot junctions of the thermopile.

For semiconductor materials presently in use Θ lies in the 0.5–1 range. It is shown in [2] that the limiting value of the amplitude a of the fluctuations being reproduced must be at least an order of magnitude smaller. Therefore, the solution of Eq. (1) for $\nu(\mathrm{Fo})$ can be sought as an expansion in terms of the small parameter a:

$$\nu(\mathrm{Fo}) = \varphi_0(\mathrm{Fo}) + a\varphi_1(\mathrm{Fo}) + a^2\varphi_2(\mathrm{Fo}) + \ldots. \tag{5}$$

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By substituting Eqs. (3)-(5) into (1) and equating terms with the same powers of α, we obtain a system of equations for the φ . Limiting ourselves to the first three terms of the series we have

$$\varphi_0\overline{\Theta} - \xi\varphi_0^2 - 0.5\varphi_0^2 = A, \tag{6}$$

$$\alpha \sin \omega \mathrm{Fo} + \beta \cos \omega \mathrm{Fo} = \varphi_1\overline{\Theta} - \varphi_0 \sin \omega \mathrm{Fo} - 2\xi\varphi_0\varphi_1 - 2\varphi_0 \int_{-\infty}^{\mathrm{Fo}} K(\mathrm{Fo} - \tau)\, \varphi_1 d\tau, \tag{7}$$

$$\varphi_2\overline{\Theta} - \varphi_1 \sin \omega \mathrm{Fo} = \xi(\varphi_1^2 + 2\varphi_0\varphi_2) + \int_{-\infty}^{\mathrm{Fo}} K(\mathrm{Fo} - \tau)(\varphi_1^2 + 2\varphi_0\varphi_2)\, d\tau. \tag{8}$$

The value of φ_0 determined from Eq. (6) is the current density which ensures the steady temperature drop $\Theta_1 - \overline{\Theta}$ across the thermopile:

$$\varphi_0 = \frac{\overline{\Theta} - \sqrt{\overline{\Theta}^2 - 2A(1 + 2\xi)}}{1 + 2\xi}. \tag{9}$$

We find the functions $\varphi_1(\mathrm{Fo})$ and $\varphi_2(\mathrm{Fo})$ in the form

$$\varphi_1(\mathrm{Fo}) = \alpha_1 \sin \omega \mathrm{Fo} + \beta_1 \cos \omega \mathrm{Fo}, \quad \varphi_2(\mathrm{Fo}) = c + \alpha_2 \sin 2\omega \mathrm{Fo} + \beta_2 \cos 2\omega \mathrm{Fo}. \tag{10}$$

The constant and variable components of the current $\alpha^2\varphi_2$ make it possible to compensate the Joule heating due to the current $\alpha\varphi_1$. The Peltier heat release resulting from the interaction of the variable component of the current and the temperature fluctuations is compensated also.

In determining the parameters of the functions φ it is necessary to use the relations

$$\begin{aligned} &\int_{-\infty}^{\mathrm{Fo}} K(\mathrm{Fo} - \tau) \sin n\omega\tau d\tau = P_n \sin n\omega \mathrm{Fo} - Q_n \cos n\omega \mathrm{Fo}, \\ &\int_{-\infty}^{\mathrm{Fo}} K(\mathrm{Fo} - \tau) \cos n\omega\tau d\tau = P_n \cos n\omega \mathrm{Fo} + Q_n \sin n\omega \mathrm{Fo}, \end{aligned} \tag{11}$$

where

$$P_n = 4\pi^2 \sum_{k=1}^{\infty} \frac{(2k-1)^2}{(2k-1)^4\pi^4 + n^2\omega^2}; \quad Q_n = 4n\omega \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4\pi^4 + n^2\omega^2}.$$

By substituting (10) into (7) and (8) we find the coefficients of the first and second harmonics of the current:

$$\alpha_1 = \frac{u_1(\omega)[\alpha(\omega) + \varphi_0] + 2\varphi_0 Q_1(\omega)\beta(\omega)}{u_1^2(\omega) + 4\varphi_0^2 Q_1^2(\omega)}, \tag{12}$$

$$\beta_1 = \frac{u_1(\omega)\beta(\omega) - 2\varphi_0 Q_1(\omega)[\alpha(\omega) + \varphi_0]}{u_1^2(\omega) + 4\varphi_0^2 Q_1^2(\omega)}. \tag{13}$$

Here

$$u_n(\omega) = \overline{\Theta} - 2\varphi_0[P_n(\omega) + \xi].$$

The ratio of the amplitudes A_1 of the first harmonics of the current and temperature and also the phase shift ψ_1 are calculated from the obvious relations

$$A_1 = \sqrt{\alpha_1^2 + \beta_1^2}, \quad \mathrm{tg}\,\psi_1 = \frac{\beta_1}{\alpha_1}.$$

The parameters of the second term of expansion (5) are

$$c = \frac{A_1^2(1+2\xi)+2\alpha_1}{4[\bar{\Theta}-\varphi_0(1+2\xi)]}, \tag{14}$$

$$\alpha_2 = \frac{u_2(\omega)X(\omega)+2\varphi_0 Q_2(\omega)Y(\omega)}{u_2^2(\omega)+4\varphi_0^2Q_2^2(\omega)}, \tag{15}$$

$$\beta_2 = \frac{u_2(\omega)Y(\omega)-2\varphi_0 Q_2(\omega)X(\omega)}{u_2^2(\omega)+4\varphi_0^2Q_2^2(\omega)}. \tag{16}$$

Here

$$X(\omega) = \alpha_1\beta_1(\xi+P_2)-0.5[(\alpha_1^2-\beta_1^2)Q_2-\beta_1]; \tag{17}$$

$$Y(\omega) = -\alpha_1\beta_1 Q_2-0.5[(\alpha_1^2-\beta_1^2)(\xi+P_2)+\alpha_1]. \tag{18}$$

The amplitude of the second harmonic A_2 and the initial phase ψ_2 can be found by starting from Eqs. (15) and (16).

Equations (9), (12)-(18) determine the current to within a term $O(a^3)$. In using the solution in practice it is of course necessary to be sure that series (5) converges, taking account of the estimate of the upper bound of the temperature amplitudes being reproduced.

Calculations with the equations presented require the determination of the sums of the series $R(\omega)$, $S(\omega)$, $P_n(\omega)$, and $Q_n(\omega)$. None of the handbooks familiar to the authors contain expressions for these sums. By using a method based on the theory of residues [3], the following expressions were obtained for these sums:

$$R = \frac{\gamma(\operatorname{sh}2\gamma-\sin 2\gamma)}{2(\operatorname{sh}^2\gamma+\sin^2\gamma)}, \tag{19}$$

$$S = \frac{\gamma(\operatorname{sh}2\gamma+\sin 2\gamma)}{2(\operatorname{sh}^2\gamma+\sin^2\gamma)}-1, \tag{20}$$

$$P_n = \frac{\operatorname{sh}\gamma\sqrt{n}+\sin\gamma\sqrt{n}}{4\gamma\sqrt{n}\left(\operatorname{sh}^2\frac{\gamma\sqrt{n}}{2}+\cos^2\frac{\gamma\sqrt{n}}{2}\right)}, \tag{21}$$

$$Q_n = \frac{\operatorname{sh}\gamma\sqrt{n}-\sin\gamma\sqrt{n}}{4\gamma\sqrt{n}\left(\operatorname{sh}^2\frac{\gamma\sqrt{n}}{2}+\cos^2\frac{\gamma\sqrt{n}}{2}\right)}. \tag{22}$$

Here $\gamma = \sqrt{\omega/2}$.

As $\omega \to 0$, $Q_n = R = S = 0$ and $P_n = 0.5$. For $\omega > 30$, the functions P_1 and Q_1 practically coincide with the asymptotic expression $1/\sqrt{2\omega}$, and the functions R and S with the expressions $\sqrt{\omega/2}$ and $\sqrt{\omega/2}-1$, respectively. The functions R and S are monotonically increasing, P is monotonically decreasing, and Q has a maximum at $\sqrt{n\omega/2} \approx 2.2$ or $\omega = 9.7/n$ ($Q_{max} \approx 0.21$).

Let us estimate the limiting values of the variable component of the current ν for small and large values of the frequency of the fluctuations of the temperature $\Theta^{(\circ)}$.

If $\omega \to 0$, $\alpha = 1 + Bi$, $\beta = 0$, $P_n = 0.5$, $Q_n = S = R = 0$, and $u_n = \bar{\Theta} - \varphi_0(2\xi + 1) = u$. Consequently,

$$\alpha_1 = \frac{\alpha+\varphi_0}{u},\ \beta_1 = 0,\ c = \alpha_1\frac{2+\alpha_1(2\xi+1)}{4u},\ \alpha_2 = 0,$$

$$\beta_2 = -\frac{\alpha_1}{2u}[1+\alpha_1(0.5+\xi)].$$

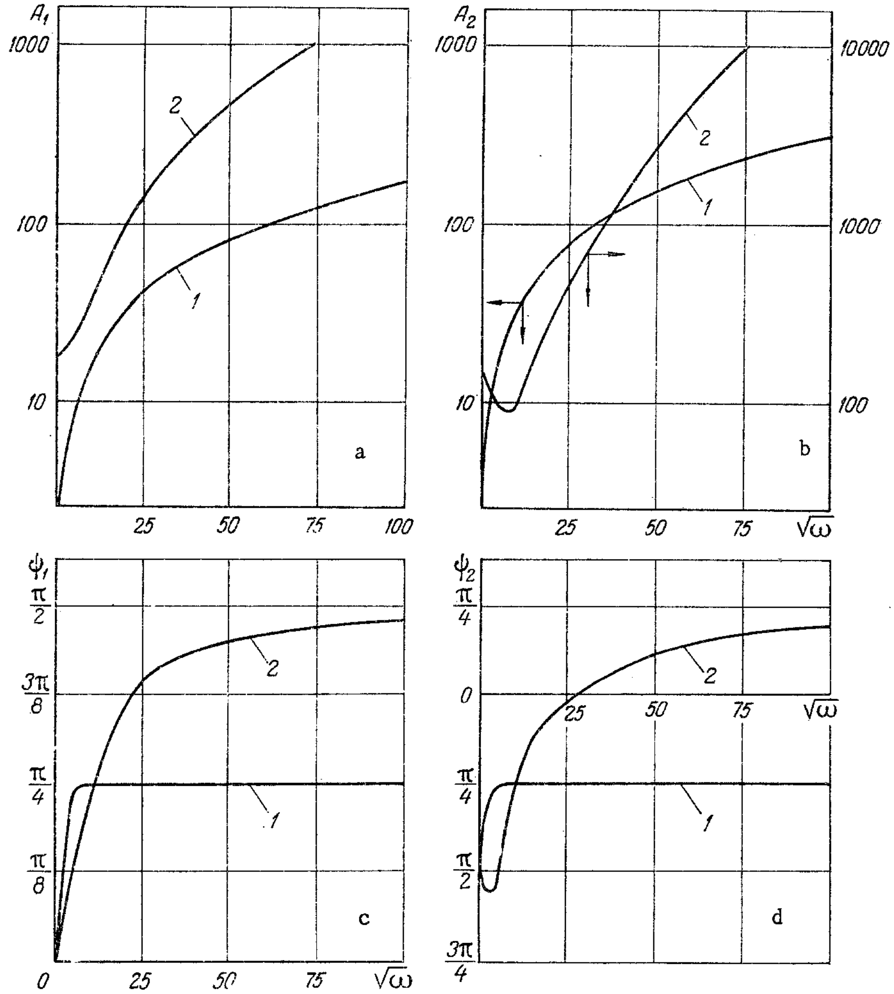

Fig. 1. Dependence of supply current parameters on the dimensionless frequency of temperature fluctuations: a) $A_1(\omega)$; b) $A_2(\omega)$; c) $\psi_1(\omega)$; d) $\psi_2(\omega)$; 1) $\eta = Bi = \xi = 0$; 2) $\eta = 0.1$, $Bi = 10$, $\xi = 0.01$.

It follows from these relations that for small ω the fluctuations of the temperature drop between junctions and the variations of the fundamental frequency of the current are in phase. The initial phase of the fluctuations of the second harmonic of the current is shifted by an angle of $\pi/2$.

For large ω there are two limiting cases:

a) no load in the form of heat capacity ($\eta = 0$); then

$$\alpha = \beta = \sqrt{\frac{\omega}{2}},\quad u_n = \bar{\Theta} - 2\varphi_0\xi = u^{(1)},\quad \alpha_1 = \beta_1 = \frac{1}{u^{(1)}}\sqrt{\frac{\omega}{2}},$$

$$c = \frac{\omega}{4u^{(1)^2}} \cdot \frac{1+2\xi}{\bar{\Theta} - \varphi_0(1+2\xi)},\quad \operatorname{tg}\psi_1 \to 1;$$

b) a load in the form of heat capacity ($\eta \neq 0$); then

$$\alpha = \sqrt{\frac{\omega}{2}},\quad \beta = \eta\omega,\quad \alpha_1 = \frac{1}{u^{(1)}}\sqrt{\frac{\omega}{2}},\quad \beta_1 = \frac{\eta\omega}{u^{(1)}},$$

$$\operatorname{tg}\psi_1 = \eta\sqrt{2\omega} \to \infty,\quad c = \frac{\eta^2\omega^2(1+2\xi)}{4u^{(1)^2}[\bar{\Theta} - \varphi_0(1+2\xi)]}.$$

Thus, the amplitude of the fundamental harmonic of the current for higher frequencies increases either proportionally to $\sqrt{\omega}$ ($\eta = 0$) or ω ($\eta \neq 0$). The current fluctuations lead the temperature fluctuations either by $\pi/4$ or $\pi/2$. The shift of the middle component of the current increases proportionally to the first or second power of the frequency.

We improve the estimate for the upper bound of the amplitude a as a function of the frequency ω. According to [2] when $\Theta_0 = \bar{\Theta} - \Theta_1$ the maximum value of a is determined by the inequality

$$a < 0.5(\Theta_0 - a)^2 (1 + 2\xi)^{-1} [(\eta\omega + R(\omega))^2 + (1 + \mathrm{Bi} + S(\omega))^2]^{-\frac{1}{2}}. \tag{23}$$

As $\omega \to 0$, this inequality coincides with the condition restricting the value of the steady temperature drop $\Delta\Theta_{max}$ for optimum current: $a < \Theta_{max}$. As ω increases, the limiting value of a decreases monotonically; for $\omega > 30$, using the asymptotic expressions for R and S

$$a < 0.5\Theta_0^2 (1 + 2\xi)^{-1} \left[\left(\eta\omega + \sqrt{\frac{\omega}{2}}\right)^2 + \left(\mathrm{Bi} + \sqrt{\frac{\omega}{2}}\right)^2\right]^{-\frac{1}{2}}. \tag{24}$$

Let us determine the limiting values of ω to ensure temperature fluctuations of more than 1°. If $z = 2 \cdot 10^{-3}$ deg^{-1}, $a \geqslant 2 \cdot 10^{-3}$. For $\Theta_0 = 0.6$ and no heat load ($\xi = \eta = \mathrm{Bi} = 0$) this requirement is satisfied for $\omega \leqslant 8100$. If $\eta = 0.1$, $\mathrm{Bi} = 10$, $\xi = 0.01$, $\omega \leqslant 670$.

Let us estimate the practicable limiting frequency f of the temperature fluctuations. Since $2\pi f t = \omega \mathrm{Fo} = \omega \varkappa t/d^2$, $f = \omega\varkappa/2\pi d^2$. According to [4] the value of the thermal diffusivity of thermoelectric materials based on Bi_2Te_3 is $(6\text{–}8) \cdot 10^{-3}$ cm^2/sec. For d = 0.3 cm, the limiting frequencies of fluctuations for the conditions considered will be 80 and 7 Hz. Temperature fluctuations of considerably higher frequencies can be obtained by using film thermopiles with a small value of d.

Figure 1 shows data characterizing the supply current parameters as functions of the specified frequency of the temperature fluctuations.

NOTATION

Dimensional quantities: T, absolute temperature; t, time; d, thickness of thermopile; j, current density; λ, c, $\varkappa$, ρ, thermal conductivity, volumetric heat capacity, thermal diffusivity, and resistivity of material of thermocouple arms, respectively; e, thermopower $z = e^2/\rho\lambda$, thermoelectric figure of merit; α, convective heat-transfer coefficient; g, heat capacity of object being cooled; r, contact resistance per unit area. Dimensionless quantities: $\Theta = zT$, temperature; $\nu = ejd/\lambda$, current density; $\mathrm{Bi} = \alpha d/\lambda$, Biot number; $\mathrm{Fo} = \varkappa t/d^2$, Fourier number; $\omega = 2\pi f d^2/\varkappa$, frequency; $\eta = g/cd$, heat capacity of load; $\xi = r/\rho d$, contact resistance.

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